5-Feb

Wednesday, February 5, 2025 1:53 PM

Generalize product $A\vec{x}$:

if column vectors of $n \times m$ matrix A are $\vec{v}_1, \dots, \vec{v}_m$ and $\vec{x}$ is a vector in $\mathbb{R}^m$ with components $x_1, \dots, x_m$

then $A\vec{x} = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_m\vec{v}_m$

ex. $A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} \overset{\text{using def'n above}}{=} 2\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} - 4\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$

$$= \begin{bmatrix} 2 \\ 8 \\ 14 \end{bmatrix} - \begin{bmatrix} 8 \\ 20 \\ 32 \end{bmatrix} + \begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 8 + 6 \\ 8 - 20 + 12 \\ 14 - 32 + 18 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

result: zero vector

Linear Combinations

a vector $\vec{b}$ in $\mathbb{R}^n$ is a linear combination of vectors $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$

if there exists scalars $x_1, \dots, x_m$ s.t.

$\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_m\vec{v}_m$

e. is $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ a linear combination of $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$?

setup:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = x\underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\vec{v}} + y\underbrace{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}}_{\vec{w}}$$

goal: find scalars x, y that will make system consistent

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ 2x \\ 3x \end{bmatrix} + \begin{bmatrix} 4y \\ 5y \\ 6y \end{bmatrix}$$

↓ can also be written as the system:

$\downarrow$ can also be written as the system:

$$\left|\begin{array}{l} x+4y=1 \\ 2x+5y=1 \\ 3x+6y=1 \end{array}\right|$$

reduce $M = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 5 & 1 \\ 3 & 6 & 1 \end{bmatrix} \xrightarrow{\text{rref}(M)} \begin{array}{c} \begin{array}{ccc} x & y & \end{array} \\ \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ $\quad x = -\frac{1}{3}$, $y = \frac{1}{3}$

conclusion: system is consistent when $x = -\frac{1}{3}$ and $y = \frac{1}{3}$

$\vec{b}$ is a linear combination of $\vec{v}$ and $\vec{w}$ when

$\vec{b} = -\frac{1}{3}\vec{v} + \frac{1}{3}\vec{w}$

Algebraic Rules for $A\vec{x}$:

if
- A is an $n \times m$ matrix
- $\vec{x}, \vec{y}$ are vectors in $\mathbb{R}^m$
- k is a scalar

then $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$
and
$A(k\vec{x}) = k \cdot A\vec{x}$

ex. consider system
$$\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left|\begin{array}{l} 3x_1 + x_2 = 7 \\ x_1 + 2x_2 = 4 \end{array}\right| \end{array}$$

$\downarrow$ think of this geometrically ...

$3x + y = 7$ and $x + 2y = 4$

also $(y = -3x + 7)$ $\quad$ $(y = -\frac{1}{2}x + 2)$

note: intersection of 2 lines or solution of system is $(2, 1)$

same (ofc)

write as an augmented matrix:

$$A = \left[\begin{array}{cc|c} 3 & 1 & 7 \\ 1 & 2 & 4 \end{array}\right] \longrightarrow \text{rref}(A) = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

alternatively, write system in <u>vector</u> form:

$$\begin{bmatrix} 3x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix}3x_1+x_2\\ x_1+2x_2\end{bmatrix}=\begin{bmatrix}7\\4\end{bmatrix}$$

$$\begin{bmatrix}3x_1\\ x_1\end{bmatrix}+\begin{bmatrix}x_2\\ 2x_2\end{bmatrix}=\begin{bmatrix}7\\4\end{bmatrix}$$

$$\begin{bmatrix}3\\1\end{bmatrix}x_1+\begin{bmatrix}1\\2\end{bmatrix}x_2=\begin{bmatrix}7\\4\end{bmatrix}$$

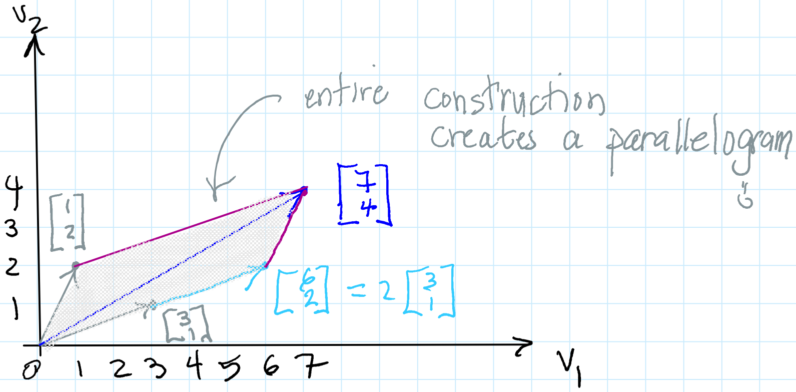
conclusion: solving system amounts to writing $\begin{bmatrix}7\\4\end{bmatrix}$ as a linear combination of vectors $\begin{bmatrix}3\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$

plotting vectors   vectors have direction and magnitude ← length

$$\vec{v}=\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

ex. plot $\begin{bmatrix}3\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$

ex plot $\begin{bmatrix}7\\4\end{bmatrix}$



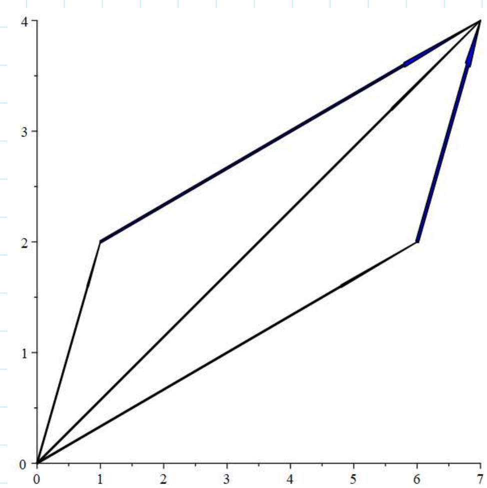
know this system works when $x_1=2$ and $x_1=1$: $\longrightarrow$ $\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$ ← scalars (components)

↑ solution set of LC

$$\begin{bmatrix}3\\1\end{bmatrix}2+\begin{bmatrix}1\\2\end{bmatrix}1=\begin{bmatrix}7\\4\end{bmatrix}$$

then $\begin{bmatrix}6\\2\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}7\\4\end{bmatrix}$

$$\begin{bmatrix}6+1\\2+2\end{bmatrix}=\begin{bmatrix}7\\4\end{bmatrix}$$



Section 2.1 Linear Transformations

ex. encoding a position

actual position (measured w Eastern longitude and Northern longitude) actual position (measured w Eastern longitude and Northern longitude)

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

get A

encoded position: $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ where $y_1 = x_1 + 3x_2$, $y_2 = 2x_1 + 5x_2$

then coding can be written as a <u>transformation</u>:

$$\vec{y} = A\,\vec{x}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

if actual position is $\vec{x} = \begin{bmatrix} 5 \\ 42 \end{bmatrix}$ then encoded position is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 42 \end{bmatrix}$$

$$= 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 42 \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 + 42(3) \\ 5(2) + 42(5) \end{bmatrix} = \begin{bmatrix} 131 \\ 220 \end{bmatrix}$$

$\uparrow \vec{y}$ (encoded position)

if given $\vec{y} = \begin{bmatrix} 133 \\ 223 \end{bmatrix}$, find actual position, $\vec{x}$

set up: $\begin{bmatrix} 133 \\ 223 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leftarrow \vec{x}$

to solve for $\vec{x}$ (in general)

$$\text{I} \left| \begin{aligned} x_1 + 3x_2 &= y_1 \\ 2x_1 + 5x_2 &= y_2 \end{aligned} \right.$$

$$\downarrow$$

$$\text{I} \left| \begin{aligned} x_1 + 3x_2 &= y_1 \\ x_2 &= 2y_1 - y_2 \end{aligned} \right.$$

$$\downarrow$$

$$\left| \begin{aligned} x_1 &= -5y_1 + 3y_2 \\ x_2 &= 2y_1 - y_2 \end{aligned} \right.$$

$$\begin{array}{c|cccc} & x_1 & x_2 & y_1 & y_2 \\ \hline *-2 & -2 & -6 & -2 & \\ & 2 & 5 & & 1 \\ \hline & 0 & -1 & -2 & 1 \\ *-1 & 0 & 1 & 2 & -1 \\ & 1 & 3 & 1 & 0 \\ & & -3 & -6 & +3 \\ \hline & 1 & 0 & -5 & 3 \end{array}$$

$$\begin{vmatrix} x_1 = -5y_1 + 3y_2 \\ x_2 = 2y_1 - y_2 \end{vmatrix} \quad 1 \quad 0 \quad -5 \quad 3$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\vec{x} = B \vec{y}$$

then matrix B is the inverse of matrix A

or $B = A^{-1}$